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SUMMATION OF CERTAIN INFINITE SERIES.

By W. J. GREENSTREET, M. A., F. R. A. S., Editor of The Mathematical Gazette, Stroud, England.

[Concluded from February Number.]

(*l*) To get a series such has $\Sigma\Sigma\frac{1}{(2m+1)^2(2n+1)^2}$, where m and n may have any integral values from 1 to ∞, we proceed as follows:

$$\left[\sum_{n=1}^{n=\infty}\frac{1}{(2n-1)^2}\right]^2=\sum_{n=1}^{n=\infty}\frac{1}{(2n-1)^4}+2\,\Sigma\Sigma\frac{1}{(2p+1)^2\,(2q+1)^2}.$$

$\therefore \Sigma\Sigma\frac{1}{(2p+1)^2\,(2q+1)^2}$ (where $2p+1$ and $2q+1$ are any odd numbers from 1 to ∞) $=\frac{1}{2}\left[\left(\frac{\pi^2}{8}\right)^2-\frac{\pi^4}{96}\right]$ by (*g*) and (*h*) $=\frac{\pi^4}{384}$.

(*m*) Consider the identity $\frac{1}{n^2(n+1)^2}\equiv\frac{1}{n^2}+\frac{1}{(n+1)^2}-2\left(\frac{1}{n}-\frac{1}{n+1}\right)$.

Here $\sum_{n=1}^{n=\infty}\frac{1}{n^2(n+1)^2}=\sum_{n=1}^{n=\infty}\frac{1}{n^2}+\left[\sum_{n=1}^{n=\infty}\frac{1}{n^2}-1\right]-2(1-\frac{1}{2}+\frac{1}{2}-\frac{1}{3}+\frac{1}{3}-\frac{1}{4}+\ldots)$

$$=\frac{\pi^2}{6}+\frac{\pi^2}{6}-1-2 \text{ by } (a) =\frac{\pi^2}{3}-3.$$

(*n*) Similarly, $\frac{1}{n^3(n+1)^3}\equiv\frac{1}{n^3}-\frac{1}{(n+1)^3}-\frac{3}{n^2(n+1)^2}$ gives

$$\sum_{n=1}^{n=\infty}\frac{1}{n^3(n+1)^3}=\left(\frac{1}{1^3}+\frac{1}{2^3}+\frac{1}{3^3}+\ldots\right)-\left(\frac{1}{2^3}+\frac{1}{3^3}+\ldots\right)-3\sum_{n=1}^{n=\infty}\frac{1}{n^2(n+1)^2}$$

$$=1-3\left(\frac{\pi^2}{3}-3\right)=10-\pi^2, \text{ by } (m).$$

(*o*) In like manner, $\frac{1}{n^4\,(n+1)^4}\equiv\frac{1}{n^4}+\frac{1}{(n+1)^4}-\frac{2}{n^2\,(n+1)^2}-\frac{4}{n^3\,(n+1)^3}$

gives $\sum_{n=1}^{n=\infty}\frac{1}{n^4\,(n+1)^4}=\sum_{n=1}^{n=\infty}\frac{1}{n^4}+\left[\sum_{n=1}^{n=\infty}\frac{1}{n^4}-1\right]-2\sum_{n=1}^{n=\infty}\frac{1}{n^2\,(n+1)^2}$

$$-4\sum_{u=1}^{n=\infty}\frac{1}{n^3\,(n+1)^3}=2.\frac{\pi^4}{90}-1-2\left(\frac{\pi^2}{3}-3\right)-4(10-\pi^2),\text{ by }(c),\ (m),\ (n)$$

$$=\frac{\pi^4}{45}+\frac{10\pi^2}{3}-35.$$

(*p*) The type $\frac{1}{1.2.3},\ \frac{1}{2.3.4},\ \frac{1}{3.4.5},\ \ldots$ suggests the identity

$$\frac{1}{n\,(n+1)\,(n+2)}\equiv\tfrac{1}{2}.\frac{1}{n}-\frac{1}{n+1}+\tfrac{1}{2}.\frac{1}{n+2}.$$

Squaring, we have

$$\frac{1}{[n(n+1)\,(n+2)]^2}\equiv\tfrac{1}{4}.\frac{1}{n^2}+\frac{1}{(n+1)^2}+\tfrac{1}{4}.\frac{1}{(n+2)^2}-\tfrac{3}{4}\left(\frac{1}{n}-\frac{1}{n+2}\right).$$

Hence $\sum_{n=1}^{n=\infty}\frac{1}{[n(n+1)\,(n+2)]^2}=\tfrac{1}{4}.\frac{\pi^2}{6}+\left(\frac{\pi^2}{6}-1\right)+\tfrac{1}{4}\left(\frac{\pi^2}{6}-1-\frac{1}{2^2}\right)-\tfrac{3}{4}.\tfrac{3}{2}$ by (*a*)

$$=\tfrac{1}{4}\pi^2-\tfrac{39}{16}.$$

(*q*) Since $\frac{\sin\theta}{\theta}=\prod_{n=1}^{n=\infty}\left(1-\frac{\theta^2}{n^2\,\pi^2}\right)$, taking logarithms and differentiating, we have

$$\frac{\cos\theta}{\sin\theta}-\theta=-\frac{2\theta}{\pi^2}\sum_{n=1}^{n=\infty}\frac{1/n^2}{1-(\theta^2/n^2\pi^2)}.$$

$$\therefore\ \frac{\theta\sin\theta-\cos\theta}{\sin\theta}=\frac{2\theta^2}{\pi^2}\left[\frac{1}{1^2}\left(1+\frac{\theta^2}{1^2\,\pi^2}+\frac{\theta^4}{1^4\,\pi^4}+\ldots\right)\right.$$

$$\left.+\frac{1}{2^2}\left(1+\frac{\theta^2}{2^2\,\pi^2}+\frac{\theta^4}{2^4\,\pi^4}+\ldots\right)+\ldots\right]$$

$$=2\left[\frac{\theta^2}{\pi^2}.\Sigma_2+\frac{\theta^4}{\pi^4}\,\Sigma_4+\frac{\theta^6}{\pi^6}\,\Sigma_6+\ldots\right].$$

$$\therefore\ \frac{\theta\sin\theta-\cos\theta}{2}=\sin\theta\left[\frac{\theta^2}{\pi^2}\,\Sigma_2+\frac{\theta^4}{\pi^4}\,\Sigma_4+\ldots\right]$$

$$=\left(\theta-\frac{\theta^3}{3!}+\frac{\theta^5}{5!}-\ldots\right)\left[\frac{\theta^2}{\pi^2}\,\Sigma_2+\frac{\theta^4}{\pi^4}\,\Sigma_4+\ldots\right].$$

Now equating the coefficients of θ^{2n+1}, we have, on expanding $\theta\sin\theta-\cos\theta$,

$$\frac{n\pi^{2n}}{(2n+1)!}=\Sigma_2.\frac{\pi^{2n-2}}{(2n-1)!}-\Sigma_4.\frac{\pi^{2n-4}}{(2n-3)!}+\ldots-(-1)^n\Sigma_{2n},$$

where $\Sigma_n=\frac{1}{1^n}+\frac{1}{2^n}+\frac{1}{3^n}+\ldots$ to ∞.

Hence for $n=1$, $\Sigma_2=\frac{\pi^2}{3!}$, which is (*a*);

$$n=2,\ \Sigma_2.\frac{\pi^2}{3!}-\Sigma_4=\frac{2\pi^4}{5!}\text{ and }\Sigma_4=\frac{\pi^4}{90},\text{ which is (c);}$$

$$n=3,\ \Sigma_2.\frac{\pi^4}{5!}-\Sigma_4.\frac{\pi^2}{3!}+\Sigma_6.\frac{1}{1!}=\frac{3\pi^6}{7!}.$$

$$\therefore\Sigma_6=\frac{-\pi^6}{3!\,5!}+\frac{\pi^6}{90.3!}+\frac{3\pi^6}{7!}=\frac{\pi^6}{945}.$$

(*r*) If 1, *a*, *b*, *c*, ..., be all the prime numbers, the numbers not divisible by the square of any prime (unity excepted) are *a*, *b*, *c*, ..., *ab*, *ac*, ..., *abc*, ... Then, the sum of the squares of the reciprocals of all numbers not divisible by the square of any prime is

$$1+a^{-2}+b^{-2}+c^{-2}+\ldots+a^{-2}b^{-2}+\ldots+a^{-2}b^{-2}c^{-2}+\ldots=\Pi(1+a^{-2})=15/\pi^2\text{ by (e).}$$

(*s*) *Lemma.* To find the limit of $\frac{\sin\theta}{1-(\theta^2/\pi^2)}$ when $\theta=\pi$. Put $\theta=\pi+h$.

$$\frac{\sin\theta}{1-(\theta^2/\pi^2)}=\frac{\sin h}{(2\pi h+h^2)/\pi^2}=\frac{\sin h}{h}.\frac{1}{2/\pi+h/\pi^2}=\tfrac{1}{2}\pi\text{ for }h=0.$$

Now $\frac{\sin\theta}{\theta}=\prod\limits_{n=1}^{n=\infty}\left(1-\frac{\theta^2}{n^2\,\pi^2}\right)$. On dividing both sides by all factors of the type $1-\frac{\theta^2}{\lambda^2\pi^2}$ where λ is a prime, we have

$$\frac{\sin\theta}{\theta[1-(\theta^2/\pi^2)]\,[1-(\theta^2/2^2\pi^2)]\ldots}=\left(1-\frac{\theta^2}{4^2\pi^2}\right)\left(1-\frac{\theta^2}{6^2\pi^2}\right)\ldots$$

Now if $\theta=\pi$ this becomes

$$\tfrac{1}{2}\left(1-\frac{1}{2^2}\right)^{-1}\left(1-\frac{1}{3^2}\right)^{-1}\ldots=\left(1-\frac{\theta^2}{4^2\pi^2}\right)\left(1-\frac{\theta^2}{6^2\pi^2}\right)\ldots$$

The left hand$=\frac{1}{2}\ (1+\frac{1}{2^2}+\frac{1}{2^4}+\ldots)\ (1+\frac{1}{3^2}+\frac{1}{3^4}+\ldots)\ldots$

$$=\tfrac{1}{2}\ (1+\frac{1}{2^2}+\frac{1}{3^2}+\ldots=\frac{\pi^2}{12}\text{ by }(a).$$

Taking logarithms of both sides

$$2\log\pi-\log 12=\frac{1}{4^2}+\frac{1}{6^2}+\frac{1}{8^2}+\ldots+\tfrac{1}{2}\left(\frac{1}{4^4}+\frac{1}{6^4}+\ldots\ \right)+\ \tfrac{1}{3}\left(\frac{1}{4^6}+\frac{1}{6^6}+\ldots\ \right)+\ldots$$

$$=\Sigma_2+\tfrac{1}{2}\,\Sigma_4+\tfrac{1}{3}\,\Sigma_6+\ldots$$

where S_r is the sum of the reciprocals of the rth powers of all numbers that are not prime.

(t) Equate coefficients of θ^8 in (A)

$$-\frac{1}{4\pi^8}\ \Sigma_8=\frac{1}{9!}-\tfrac{1}{2}\left(\frac{1}{5!\,5!}+\frac{2}{3!\,7!}\right)+\tfrac{1}{3}.\frac{3}{3!\,3!\,5!}-\tfrac{1}{4}\frac{1}{(3\,!)^4}=-\ \tfrac{1}{4}.\frac{1}{9450}.$$

$$\therefore\ \Sigma_8=\frac{\pi^8}{9450}.$$

(u) $$\sum_{n=1}^{n=\infty}(-1)^{n-1}\frac{1}{n^4}\left\{\sum_{n=1}^{n=\infty}\frac{1}{n^4}\right\}\left(1-2.\frac{1}{2^4}\right)=\frac{\pi^4}{90}.\frac{7}{8}=\frac{7\pi^4}{720}.$$

(v) $$\sum_{n=1}^{n=\infty}(-1)^{n-1}\frac{1}{n^8}\left(\sum_{n-1}^{n=\infty}\frac{1}{n^8}\right)\left(1-2.\frac{1}{2^8}\right)=\frac{7\pi^4}{720}\times\frac{127}{128}.$$

(w) $$\sum_{n=1}^{n=\infty}\frac{1}{(2n-1)^8}=\left(\sum_{n=1}^{n=\infty}\frac{1}{n^8}\right)\left(1-\frac{1}{2^8}\right)=\frac{\pi^8}{9450}\times\frac{255}{256}=\frac{17\pi^8}{256\times 630}.$$

(x) $$\sum_{n=1}^{n=\infty}\frac{4}{n^2\ (n+3)^2}=\tfrac{4}{9}\sum_{n=1}^{n=\infty}\left(\frac{1}{n^2}+\frac{1}{(n+3)^2}\right)$$

$$-\ \tfrac{8}{9}\sum_{n=1}^{n=\infty}\left(\frac{1}{n(n+1)}-\frac{2}{n(n+1)\,(n+2)}+\frac{2}{n(n+1)\,(n+2)\,(n+3)}\right)$$

$$=\tfrac{4}{9}\left(\frac{\pi^2}{6}+\frac{\pi^2}{6}-\frac{1}{1^2}-\frac{1}{2^2}-\frac{1}{3^2}\right)-\ \tfrac{8}{9}\left(1-\frac{2}{4}+\frac{2}{18}\right)=\frac{4\pi^2-31}{27}.$$

(y) $$\frac{2}{1^3+2^3}+\frac{3}{3^3+4^3+5^3}+\frac{4}{6^3+7^3+8^3+9^3}+\ldots$$

$$=\sum_{n=1}^{n=\infty}\frac{8}{n^2\,(n+1)^2\ (n+2)^2}$$

$$=\sum_{n=1}^{n=\infty}\left(\frac{2}{n^2}+\frac{8}{(n+1)^2}+\frac{2}{(n+2)^2}-\frac{12}{n(n+1)}+\frac{12}{n(n+1)(n+2)}\right)$$

$$=\frac{12\pi^2}{6}-8-2\left(\frac{1}{1^2}+\frac{1}{2^2}\right)-12+3=2\pi^2-19\tfrac{1}{2}.$$

(*z*) $$\frac{1}{1^2.2^2}-\frac{3}{4^2.5^2}+\frac{5}{7^2.8^2}+\ldots=\tfrac{1}{3}\left[1-\frac{1}{2^2}-\frac{1}{4^2}+\frac{1}{5^2}+\frac{1}{7^2}-\frac{1}{8^2}+\ldots\right]$$

$$=\tfrac{1}{3}\left[1-\frac{1}{2^2}+\frac{1}{3^2}-\frac{1}{4^2}+\frac{1}{5^2}-\frac{1}{6^2}+\frac{1}{7^2}-\ldots\right]$$

$$-\frac{1}{3^3}\left[1-\frac{1}{2^2}+\frac{1}{3^2}-\ldots\right]=\left(\frac{1}{3}-\frac{1}{3^3}\right)\frac{\pi^2}{12}\text{ by }(i)=\frac{2\pi^2}{81}.$$

(*α*) From the identity

$$\frac{1}{(4n-3)^4\,(4n-1)^4}=\frac{1}{2^4}\left[\frac{1}{(4n-3)^4}+\frac{1}{(4n-1)^4}\right]-\frac{1}{2^3}\left[\frac{1}{(4n-3)^3}-\frac{1}{(4n-1)^3}\right]$$

$$+\frac{5}{2^5}\left[\frac{1}{(4n-3)^2}+\frac{1}{(4n-1)^2}\right]-\frac{5}{2^5}\left[\frac{1}{4n-3}-\frac{1}{4n-1}\right]$$

we have

$$\frac{1}{1^4.3^4}+\frac{1}{5^4.7^4}+\frac{1}{9^4.11^4}+\ldots=\sum_{n=1}^{n=\infty}\frac{1}{(n-3)^4(n-1)^4}$$

$$=\frac{\pi^4}{96\times2^4}-\frac{1}{2^3}\cdot\frac{\pi^3}{32}+\frac{5}{2^5}\cdot\frac{\pi^2}{8}-\frac{5}{2^5}\cdot\frac{\pi}{4}=\frac{1}{1536}(\pi^4-6\pi^3+30\pi^2-60\pi).$$

To find $\frac{1}{1^4.\,3^4}+\frac{1}{3^4.\,5^4}+\frac{1}{5^4.\,7^4}+\ldots=\sum_{n=1}^{n=\infty}\frac{1}{(2n-1)^4\,(2n+1)^4}$ we use the identity

$$\frac{1}{(2n-1)^4\,(2n+1)^4}=\frac{1}{2^4}(u^4+v^4)-\frac{1}{2^3}(u^3-v^3)+\frac{5}{2^5}(u^2-v^2)$$

$$-\frac{5}{2^5}(u-v)\text{ where }u=\frac{1}{2n-1},\ v=\frac{1}{2n+1}.$$

Hence the sum required is found to be $\frac{1}{768}(\pi^4+30\pi^2-384)$.

(*β*) Taking logarithms in (*A*) and equating coefficients of θ^4, and also of θ^8 we have $\sum_{n=1}^{n=\infty}\frac{1}{n^4}$ and $\sum_{n=1}^{n=\infty}\frac{1}{n^8}$ as in (*c*) and (*t*).

$$\text{Now }2\sum_{m=1}^{m=\infty}\frac{1}{m^4}\sum_{n=1}^{n=\infty}\frac{1}{n^4}=\left(\sum_{n=1}^{n=\infty}\frac{1}{n^4}\right)^2-\sum_{n=1}^{n=\infty}\frac{1}{n^8}$$

$$=\frac{\pi^8}{8100}-\frac{\pi^8}{9450}\text{ by }(c)\text{ and }(t)=\frac{2\pi^8}{113400}.$$

$$\therefore \sum_{m=1}^{m=\infty} \frac{1}{m^4} \sum_{n=1}^{n=\infty} \frac{1}{n^4} = \frac{384\pi^8}{9!\,5!}$$

and this is the sum of the products of the fourth powers of the reciprocals of every pair of positive integers.

(γ) Consider the identity $\frac{1}{n(n+1)(n+2)} = \frac{1}{2.n} - \frac{1}{n+1} + \frac{1}{2(n+2)}$.

We have $\frac{1}{n^2(n+1)^2(n+2)^2} = \frac{1}{4}\cdot\frac{1}{n^2} + \frac{1}{(n+1)^2} + \frac{1}{4(n+2)^2}$

$$-\frac{1}{n}\cdot\frac{1}{(n+1)} + \frac{1}{2}\cdot\frac{1}{n}\cdot\frac{1}{n+2} - \frac{1}{(n+1)\ (n+2)} = \ldots - \frac{3}{4}\left(\frac{1}{n} - \frac{1}{n+1}\right).$$

$$\therefore \sum_{n=1}^{n=\infty} \frac{1}{n^2(n+1)^2(n+2)^2} = \tfrac{1}{4}\sum_{n=1}^{n=\infty}\frac{1}{n^2} + \tfrac{1}{4}\sum_{n=1}^{n=\infty}\frac{1}{(n+1)^2} + \tfrac{1}{4}\sum_{n=1}^{n=\infty}\frac{1}{(n+2)^2}$$

$$-\tfrac{3}{4}(1-\tfrac{1}{3}+\tfrac{1}{2}-\tfrac{1}{4}+\tfrac{1}{3}-\tfrac{1}{5}+\tfrac{1}{4}-\tfrac{1}{6}-\ldots)$$

$$= \frac{\pi^2}{24} + \frac{\pi^2}{6} - 1 + \frac{1}{4}\left(\frac{\pi^2}{6} - 1 - \frac{1}{4}\right) - \frac{3}{4}\cdot\frac{3}{2}$$

$$= \frac{\pi^2}{4} - \frac{39}{16}.$$

(δ) Since, when a, b, c, ... are all the primes save unity

$\prod_{2}^{\infty} (1-a^{-8})^{-1} = \frac{\pi^8}{9450}$, and $\prod_{2}^{\infty} (1-a^{-4})^{-1} = \frac{\pi^4}{90}$ by (c) and (t)

$$\prod_{2}^{\infty} (1+a^{-4})^{-1} = \frac{\pi^4}{105},\ i.\ e.,\ \frac{2^4}{2^4+1}\cdot\frac{3^4}{3^4+1}\cdot\frac{5^4}{5^4+1}\cdots = \frac{\pi^4}{105}.$$

(ϵ) Since $\sum_{n=1}^{n=\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$, $\sum_{u=1}^{n=\infty} \frac{1}{(3n)^2} = \frac{\pi^2}{54}$.

Subtracting, we have $\frac{4\pi^2}{27} = 1 + \frac{1}{2^2} + \frac{1}{4^2} + \frac{1}{5^2} + \frac{1}{7^2} + \ldots$ where the denominators are the squares of all numbers not divisible by 3,

Euler's *Introductio in Analysis Infinitorum* (1748) Vol. I, Caput X. *De usu Factorum inventorum in definiendis summis Serierum infinitarum,* will be found a happy hunting-ground for those who want more of these series. The Latin is easily comprehended, the reader soon getting used to *Dignitates, Potestates, Exponentes,* and the like, while the methods are worth studying carefully by all interested in the historical development of mathematics.